

## Exact eigenstates of some spin-1/2 Heisenberg chains with $1/r^2$ exchange

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1992 J. Phys. A: Math. Gen. 25 5447

(<http://iopscience.iop.org/0305-4470/25/21/007>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.59

The article was downloaded on 01/06/2010 at 17:27

Please note that [terms and conditions apply](#).

# Exact eigenstates of some spin- $\frac{1}{2}$ Heisenberg chains with $1/r^2$ exchange

P J Forrester†

Department of Mathematics, La Trobe University, Bundoora, Victoria 3083, Australia

Received 24 February 1992

**Abstract.** A new derivation of the exact eigenstates and corresponding energy levels of some spin- $\frac{1}{2}$  Heisenberg chains with  $1/r^2$  exchange found by Haldane is given. This method is then applied to obtain some exact eigenstates and corresponding energy levels of a two-component generalization of Haldane's spin system.

## 1. Introduction

Haldane [1, 2] has provided some exact eigenstates and corresponding eigenvalues for the one-dimensional spin- $\frac{1}{2}$  Heisenberg chain with Hamiltonian

$$\left(\frac{\pi}{N}\right)^2 \sum_{1 \leq n < n' \leq N} \frac{1}{\sin^2 \pi(n - n')/N} (S_n^x S_{n'}^x + S_n^y S_{n'}^y + \Delta S_n^z S_{n'}^z) \quad (1.1)$$

(the case  $\Delta = 1$  has also been considered by Shastry [3]; see [4] for a discussion of related Hamiltonians). The system represented by this Hamiltonian can be thought of as having spins at  $N$  equally spaced lattice sites on a circle of circumference length  $N$ , with an exchange interaction proportional to the inverse square of the chord length.

There is a striking correspondence between the eigenstates of the Hamiltonian (1.1) found in [1] and [2] and the eigenfunctions of the continuous Schrödinger operator

$$-\sum_{j=1}^M \frac{\partial^2}{\partial(x_j)^2} + g \left(\frac{\pi}{N}\right)^2 \sum_{1 \leq j < k \leq M} \frac{1}{\sin^2 \pi(x_k - x_j)/N}. \quad (1.2)$$

Explicitly, for

$$\Delta = \frac{1}{4}g = \frac{1}{2}m(m - 1) \quad (1.3)$$

where  $m$  denotes an even positive integer, exact eigenstates of (1.1) can be obtained from exact wavefunctions of (1.2) by simply replacing the continuous variables  $x_k$

† E-mail address: MATPJF@LURE.LATROBE.EDU.AU

by the corresponding lattice site labels  $n_k$ . In [1] this fact was exhibited by explicit verification (as it is in [3], in the special case  $\Delta = 1$ ).

An alternative approach to explicit verification for revealing the correspondence between the quantum lattice Hamiltonians and continuous Schrödinger operators has been observed by Sutherland [5]. He pointed out that, under certain conditions, the Fourier transform of the kinetic energy operator and the discrete Fourier transform of the hopping operator are equal.

With  $\phi(x) = \phi(x + N)$  the former quantity is defined as

$$\frac{1}{N} \int_0^N e^{-2\pi i x k/N} \left( -\frac{\partial^2}{\partial x^2} \phi(x) \right) dx \quad (1.4a)$$

which equals

$$\left( \frac{2\pi k}{N} \right)^2 \frac{1}{N} \int_0^N e^{-2\pi i x k/N} \phi(x) dx \quad (1.4b)$$

while the latter quantity is defined as

$$\frac{1}{N} \sum_{\ell=0}^{N-1} e^{-2\pi i k \ell/N} \left( \sum_{j=0}^{N-1} V(j) \phi(\ell + j) \right) \quad (1.5a)$$

which equals

$$\left( \sum_{j=0}^{N-1} V(j) e^{2\pi i k j/N} \right) \frac{1}{N} \sum_{\ell=0}^{N-1} e^{-2\pi i k \ell/N} \phi(\ell) \quad (1.5b)$$

where  $V(j)$  is the weight of a step of  $j$  units, and we assume  $V(j) = V(j + N)$ . In both (1.4a) and (1.5a), for comparison with the results of [1], we have used  $-k$  instead of the usual  $k$  in the definitions of the transforms.

For integer  $k$ ,  $0 \leq k \leq N - 1$ , equality between (1.4) and (1.5) can be achieved by choosing

$$V(j) = \frac{1}{N} \sum_{k=0}^{N-1} (2\pi k/N)^2 e^{-2\pi i k j/N} \quad (1.6)$$

and requiring that

$$\int_0^N e^{-2\pi i x k/N} \phi(x) dx = \sum_{\ell=0}^{N-1} e^{-2\pi i k \ell/N} \phi(\ell). \quad (1.7)$$

The equation (1.7) is valid if the Fourier expansion of the eigenstate is restricted to a single Brillouin zone so that

$$\phi(x) = \sum_{p=0}^{N-1} c_p e^{2\pi i x p/N}. \quad (1.8)$$

Sutherland [5] intimated that the correspondence between the Hamiltonians (1) and (2) could be exhibited in a similar way, but no details were given. In section 2 of this paper the details of the required calculation will be provided, thus giving a new derivation of the class of eigenstates and corresponding eigenvalues given in [1-3]. The method will then be applied to the derivation of some exact eigenstates of a two-component spin- $\frac{1}{2}$  Heisenberg chain with  $1/r^2$  exchange, via its correspondence with a solvable two-component Schrödinger operator.

## 2. Derivation of the exact eigenstates

### 2.1. Correspondence between the Heisenberg chain and the Bose lattice gas

To use the wavefunctions of the Schrödinger operator (1.2) to derive eigenstates of the Heisenberg chain Hamiltonian (1.1), it is first necessary to rewrite the latter as the Hamiltonian for a 'hard-core' Bose lattice gas. By writing the Pauli spin operators in terms of the raising and lowering operators:

$$S_j^x = (a_j^\dagger + a_j)/2 \quad S_j^y = (a_j^\dagger - a_j)/2i \quad S_j^z = (a_j^\dagger a_j - \frac{1}{2}) \quad (2.1)$$

the Hamiltonian (1.1) reads

$$\sum_{1 \leq n < n' \leq N} V(n - n') [\frac{1}{2}(a_n^\dagger a_{n'} + a_n a_n') + \Delta a_n^\dagger a_n a_{n'}^\dagger a_{n'}] + \frac{1}{2} \Delta c(-M + \frac{1}{4}N) \quad (2.2a)$$

where

$$c = \sum_{j=1}^{N-1} V(j) \quad (2.2b)$$

and

$$V(j) = \left(\frac{\pi}{N}\right)^2 \frac{1}{\sin^2 \pi j/N}. \quad (2.2c)$$

In (2.2a) we have replaced the operator

$$\sum_{j=1}^N a_j^\dagger a_j \quad (2.3)$$

by its eigenvalue  $M$ ,  $0 \leq M \leq N$ . This is possible since the operator (2.3) commutes with the Hamiltonian (2.2a), a fact which can be checked by using the relations

$$[a_j, a_k] = [a_j^\dagger, a_k^\dagger] = [a_j, a_k^\dagger] = 0 \quad j \neq k \quad (2.4a)$$

$$a_j^2 = (a_j^\dagger)^2 = 0 \quad \{a_j, a_j^\dagger\} = 1. \quad (2.4b)$$

The commutation relations (2.4a) are those of a Bose lattice gas, while the relations (2.4b) express the 'hard core' condition that two particles cannot occupy the same site. The operator (2.3) counts the total number of up spins in the  $z$ -direction for the Heisenberg chain, while for the Bose lattice gas this operator gives the total number of particles.

The advantage of the representation (2.2a) is that it enables the eigenvalue problem for the eigenstate  $\psi(\ell_1, \dots, \ell_M)$  ( $\ell_j$  denotes the coordinate of the  $j$ th particle) with  $M$  particles as a difference equation. We have

$$\begin{aligned} & \frac{1}{2} \sum_{j=1}^M \sum_{p=1}^{N-1} V(p) \psi(\ell_1, \dots, \ell_{j-1}, \ell_j + p, \ell_{j+1}, \dots, \ell_M) \\ & \quad + \Delta \sum_{1 \leq j < k \leq M} V(\ell_j - \ell_k) \psi(\ell_1, \dots, \ell_M) \\ & = (\lambda + \frac{1}{2} \Delta c(M - \frac{1}{4}N)) \psi(\ell_1, \dots, \ell_M) \end{aligned} \quad (2.5)$$

where  $\lambda$  denotes the energy eigenvalue.

## 2.2. Exact wavefunctions of the $1/r^2$ many-body Schrödinger operator

We want to relate the solutions of the difference equation (2.5) to the solution of the Schrödinger equation with Hamiltonian (1.2). It is convenient to first collect together some properties of the latter.

Sutherland [6] has shown that the ground-state wavefunction of (1.2) is given by the BDJ pair product form

$$\psi_0(x_1, \dots, x_M) = \prod_{1 \leq j < k \leq M} |\sin \pi(x_k - x_j)/N|^m \quad (2.6)$$

where the parameter  $m$  and the coupling  $g$  are related by

$$g = 2m(m - 1) \quad (2.7)$$

(also, for stability, we require  $g \geq -\frac{1}{2}$ ). The excited states are given by [7]

$$\psi_0(x_1, \dots, x_M) \phi(x_1, \dots, x_M) \quad (2.8)$$

where  $\phi$  satisfies the eigenvalue equation

$$\begin{aligned} & - \sum_{j=1}^M \frac{\partial^2}{\partial(x_j)^2} \phi - 2m \left( \frac{\pi}{N} \right) \sum_{1 \leq j < k \leq M} \cot \pi(x_k - x_j)/N \left( \frac{\partial}{\partial x_k} - \frac{\partial}{\partial x_j} \right) \phi \\ & = (E - E_0) \phi. \end{aligned} \quad (2.9)$$

Here  $E$  denotes the energy of the excited states and  $E_0$  denotes the ground-state energy, which is given explicitly by [6]

$$E_0 = \left( \frac{\pi}{N} \right)^2 m^2 \frac{M(M^2 - 1)}{3}. \quad (2.10)$$

The solutions [7] of (2.9) are symmetric Laurent polynomials in the variables

$$e^{2\pi i x_\ell / N} \quad \ell = 1, \dots, M. \tag{2.11}$$

To each set of distinct integers

$$p_1 \leq p_2 \leq \dots \leq p_M \tag{2.12}$$

there corresponds a Laurent polynomial which satisfies (2.9) and has greatest order, in the variables (2.11),  $p_M$ , and least order  $p_1$ . The corresponding energy eigenvalue is

$$E = E_0 + \left(\frac{2\pi}{N}\right)^2 \sum_{j=1}^M (p_j)^2 + m \left(\frac{2\pi}{N}\right)^2 \sum_{1 \leq j < k \leq M} (p_k - p_j). \tag{2.13}$$

A further relevant feature of the solutions of (2.9) is that if we denote the wavefunction (2.8) corresponding to the integers (2.12) by

$$\psi_{p_1 \dots p_m}(x_1, \dots, x_M) \tag{2.14}$$

then

$$\psi_{p_1+J \dots p_m+J}(x_1, \dots, x_m) = \left( \prod_{j=1}^M e^{2\pi i J x_j / N} \right) \psi_{p_1 \dots p_m}(x_1, \dots, x_M). \tag{2.15}$$

Our method of establishing the relationship between the difference equation (2.5) and the solution of the Schrödinger equation with Hamiltonian (1.2) uses an identity of the form (1.7). This in turn relies on the Fourier expansion of the wavefunction (2.8) having components entirely within a single Brillouin zone, so that

$$\psi_0 \phi = \sum_{j_1=0}^{N-1} \dots \sum_{j_M=0}^{N-1} a_{j_1 \dots j_M} \prod_{\ell=1}^M e^{2\pi i j_\ell x_\ell / N}. \tag{2.16}$$

From (2.6) and the comment between (2.12) and (2.13), this is the case if and only if  $m$  is an even integer and

$$0 \leq -\frac{1}{2}m(M-1) + p_1 \tag{2.17a}$$

$$N-1 \geq \frac{1}{2}m(M-1) + p_M. \tag{2.17b}$$

2.3. *Solution of the discrete problem*

In following the procedure of Sutherland [5] given in the introduction, we first take the  $M$ -dimensional discrete Fourier transform (generalizing (1.5a)) of both sides of the difference equation (2.5). Using the result in the appendix to evaluate

$$\sum_{j=0}^{N-1} e^{-2\pi i k j / N} V(j) \tag{2.18}$$

we obtain

$$\begin{aligned} & \left(\frac{\pi}{N}\right)^2 \sum_{j=1}^M (k_j - N/2)^2 \tilde{\psi}(k_1, \dots, k_M) + \Delta S(k_1, \dots, k_M) \\ &= \left[ \lambda + \frac{\Delta c}{2} \left(M - \frac{N}{4}\right) + \frac{1}{12} \left(\frac{\pi}{N}\right)^2 M(N^2 + 2) \right] \tilde{\psi}(k_1, \dots, k_M) \end{aligned} \tag{2.19a}$$

where

$$\tilde{\psi}(k_1, \dots, k_M) := DF[\psi(\ell_1, \dots, \ell_M)] \tag{2.19b}$$

$$S(k_1, \dots, k_M) := DF \left[ \sum_{1 \leq j < k \leq M} V(\ell_k - \ell_j) \psi(\ell_1, \dots, \ell_M) \right] \tag{2.19c}$$

and

$$c = \frac{1}{3} \left(\frac{\pi}{N}\right)^2 (N^2 - 1). \tag{2.19d}$$

Here the discrete Fourier transform  $DF$  is defined as

$$DF := \prod_{j=1}^M \frac{1}{N} \sum_{\ell_j=0}^{N-1} e^{-2\pi i \ell_j k_j / N} \tag{2.20a}$$

and each  $k_j$  is an integer in the interval

$$0 \leq k_j \leq N - 1. \tag{2.20b}$$

The evaluation (2.19d) follows from the definition (2.2c) and the result in the appendix. We note that the equation (2.19a) is equivalent to (2.5) in the sense that  $\psi$  satisfies (2.5) if and only if  $\tilde{\psi}$  satisfies (2.19a).

Now consider the wavefunction (2.15) with  $J = -M/2$ . With each  $p_j$  an integer and  $N$  even the numbers  $p_j - N/2$  are integers and (2.15) is periodic under the translation  $x_j \mapsto x_j + N$ . However if  $N$  is odd, (2.15) changes sign under this

mapping. Nonetheless, in both cases (2.15) satisfies the Schrödinger equation with Hamiltonian (1.2) and, from (2.13) and (2.18), energy eigenvalue

$$E = \left(\frac{\pi}{N}\right)^2 m^2 \frac{N(N^2 - 1)}{3} + \left(\frac{2\pi}{N}\right)^2 \sum_{j=1}^M \left(p_j - \frac{N}{2}\right)^2 + m \left(\frac{2\pi}{N}\right)^2 \sum_{1 \leq j < k \leq M} (p_k - p_j). \tag{2.21}$$

Analogous to (1.4), but taking special note that (2.19a) contains  $(k_j - N/2)^2$  rather than  $k_j^2$ , we take the Fourier transform of this Schrödinger equation with the Fourier variables  $k_j$  replaced by  $k_j - N/2, j = 1, \dots, M$ . This gives

$$\begin{aligned} \left(\frac{2\pi}{N}\right)^2 \sum_{j=1}^M \left(k_j - \frac{N}{2}\right)^2 \tilde{\psi}_{p_1 \dots p_M}(k_1, \dots, k_M) + gI(k_1, \dots, k_M) \\ = E \tilde{\psi}_{p_1 \dots p_M}(k_1, \dots, k_M) \end{aligned} \tag{2.22a}$$

where

$$\tilde{\psi}_{p_1 \dots p_M}(k_1, \dots, k_M) = F[\psi_{p_1 \dots p_M}(x_1, \dots, x_M)] \tag{2.22b}$$

$$I(k_1, \dots, k_M) = F \left[ \sum_{1 \leq j < k \leq M} V(x_k - x_j) \psi_{p_1 \dots p_M}(x_1, \dots, x_M) \right] \tag{2.22c}$$

and  $E$  is given by (2.21). The Fourier transform operator  $F$  is defined by

$$F := \prod_{j=1}^M \frac{1}{N} \int_0^N dx_j e^{-2\pi i x_j k_j / N}. \tag{2.23}$$

Suppose  $m$  is even and (2.17) holds so that the wavefunction in (2.22) has a Fourier expansion of the form (2.16). Define the unknown eigenstate in (2.19) by

$$\psi(\ell_1, \dots, \ell_M) = \psi_{p_1 \dots p_M}(\ell_1, \dots, \ell_M) \tag{2.24}$$

where the right-hand side of (2.24) is the wavefunction (2.14) evaluated at the lattice coordinates of the particles. Then the discrete Fourier transform (2.19b) and the continuous Fourier transform (2.22b) are equal. Furthermore, due to the functional form (2.6) of  $\psi_0$ , if (2.14) has a Fourier expansion of the form (2.16), so too does the function in (2.22c), and, by (2.24), the function in (2.19c). Thus (2.19c) and (2.22c) are also equal, so we see that (2.22a) and (2.19a) are identical equations, provided

$$\Delta = \frac{1}{4}g = \frac{1}{2}m(m - 1) \tag{2.25}$$

and

$$\lambda = \frac{E}{4} - \frac{\Delta c}{2} \left(M - \frac{N}{4}\right) - \frac{1}{12} \left(\frac{\pi}{N}\right)^2 M(N^2 + 2) \tag{2.26}$$



where  $E$  is given by (2.21) and  $c$  by (2.19d). Hence, under the conditions stated at the beginning of this paragraph, (2.24) gives a family of eigenstates of (1.1) with corresponding energy eigenvalue (2.26). This result is contained in [2] for the special case  $\Delta = 1$ .

As a more explicit example, with  $p_1 = p_2 = \dots = p_M = J$ , the above results give that the function

$$\prod_{k=1}^M e^{2\pi i J \ell_k / N} \prod_{1 \leq j < k \leq M} \left( \sin \frac{\pi}{N} (\ell_k - \ell_j) \right)^m \tag{2.27}$$

is an eigenstate of (1.1) with  $\Delta$  given by (2.25), provided  $m$  is even and

$$0 \leq -\frac{1}{2}m(M - 1) + J \tag{2.28a}$$

$$(N - 1) \geq \frac{1}{2}m(M - 1) + J. \tag{2.28b}$$

The corresponding energy eigenvalue is

$$\begin{aligned} \lambda = & \left( \frac{\pi}{N} \right)^2 m^2 \frac{M(M^2 - 1)}{12} + \left( \frac{\pi}{N} \right)^2 M \left( J - \frac{N}{2} \right)^2 - \frac{\Delta}{6} \left( \frac{\pi}{N} \right)^2 (N^2 - 1) \left( M - \frac{N}{4} \right) \\ & - \frac{1}{12} \left( \frac{\pi}{N} \right)^2 M(N^2 + 2). \end{aligned} \tag{2.29}$$

These results agree with those given in [1] (and [3] in the special case  $\Delta = 1$ ,  $N$  even and  $J = N/2$ ).

### 3. Exact eigenstates for a two-component system

#### 3.1. Some related Hamiltonians

The method used in the previous section to obtain exact eigenstates of the Bose lattice gas Hamiltonian (2.2a) from the Schrödinger operator (1.2) can be generalized to a two-component case. Krivnov and Ovchinnikov [8] have considered the two-component generalization of (1.2):

$$\begin{aligned} - \sum_{j=1}^{M_1} \frac{\partial^2}{\partial(x_j)^2} - \frac{1}{Q} \sum_{j=1}^{M_2} \frac{\partial^2}{\partial(y_j)^2} + g_1 \sum_{1 \leq j < k \leq M_1} V(x_k - x_j) \\ + g_2 \sum_{1 \leq j < k \leq M_2} V(y_k - y_j) + g_{12} \sum_{j=1}^{M_1} \sum_{k=1}^{M_2} V(x_j - y_k) \end{aligned} \tag{3.1}$$

where  $V$  is defined by (2.2c). In this section we will use exact wavefunctions of (3.1) to obtain exact eigenstates of the two-component Bose lattice gas Hamiltonian

$$\sum_{1 \leq \ell < \ell' \leq N} V(\ell - \ell') \left[ \frac{1}{2} (a_\ell^\dagger a_{\ell'} + a_{\ell'}^\dagger a_\ell) + \Delta_{11} a_\ell^\dagger a_{\ell'} a_\ell^\dagger a_{\ell'} \right]$$

$$\begin{aligned}
 & + \sum_{1 \leq n < n' \leq QN} V((n-n')/Q) [\frac{1}{2}(\alpha_n^+ \alpha_{n'} + \alpha_n^+ \alpha_n) + \Delta_{22} \alpha_n^+ \alpha_n \alpha_{n'}^+ \alpha_{n'}] \\
 & + \Delta_{12} \sum_{\ell=1}^N \sum_{n=1}^{QN} V((\ell + \phi) - n/Q) a_\ell^+ a_\ell \alpha_n^+ \alpha_n.
 \end{aligned} \tag{3.2}$$

The system represented by the Hamiltonian (3.2) can be considered to have a circle as its domain. Species 1 is confined to  $N$  lattice sites spaced by polar angles  $2\pi/N$ , starting at angle  $2\pi\phi/N$ , while species 2 is confined to  $QN$  lattice sites spaced by polar angle  $2\pi/NQ$  starting at angle 0. The number operators for each species,

$$\sum_{\ell=1}^N a_\ell^+ a_\ell \quad \text{and} \quad \sum_{n=1}^{QN} \alpha_n^+ \alpha_n \tag{3.3}$$

commute with the Hamiltonian so we can consider a state with  $M_1, 0 \leq M_1 \leq N$ , particles of species 1 and  $M_2, 0 \leq M_2 \leq QN$ , particles of species 2.

The eigenstates then depend on the coordinates  $\ell_1, \dots, \ell_{M_1}$  of the particles of species 1 and the coordinates  $n_1, \dots, n_{M_2}$  of the particles of species 2, so they can be denoted by

$$\psi(\ell_1, \dots, \ell_{M_1}; n_1, \dots, n_{M_2}). \tag{3.4}$$

From (3.2), analogously to (2.5), the eigenvalue equation for these eigenstates can be written as a difference equation. Taking the discrete Fourier transform, the difference equation reads

$$\begin{aligned}
 & \left(\frac{\pi}{N}\right)^2 \left[ \sum_{j=1}^{M_1} (k_j - N/2)^2 + \frac{1}{Q} \sum_{j=1}^{M_2} (K_j - NQ/2)^2 \right] \tilde{\psi}(k_1, \dots, k_{M_1}; K_1, \dots, K_{M_2}) \\
 & + S(k_1, \dots, k_{M_1}; K_1, \dots, K_{M_2}) \\
 & = \left( \lambda + \frac{1}{12} \left(\frac{\pi}{N}\right)^2 [M_1(N^2 + 2) + M_2((QN)^2 + 2)/Q] \right) \\
 & \times \tilde{\psi}(k_1, \dots, k_{M_1}; K_1, \dots, K_{M_2})
 \end{aligned} \tag{3.5a}$$

where

$$\tilde{\psi} = DF_\phi[\psi(\ell_1, \dots, \ell_{M_1}; n_1, \dots, n_{M_2})] \tag{3.5b}$$

$$\begin{aligned}
 S = DF_\phi \left[ \Delta_{11} \sum_{1 \leq j < k \leq M_1} V(\ell_k - \ell_j) + \Delta_{22} \sum_{1 \leq j < k \leq M_2} V((n_k - n_j)/Q) \right. \\
 \left. + \Delta_{12} \sum_{j=1}^{M_1} \sum_{k=1}^{M_2} V((\ell_j + \phi) - n_k/Q) \right] \psi(\ell_1, \dots, \ell_{M_1}; n_1, \dots, n_{M_2}) \tag{3.5c}
 \end{aligned}$$

and the discrete Fourier transform operator is defined as

$$DF_\phi := \left( \prod_{j=1}^{N_1} \frac{1}{N} \sum_{\ell_j=0}^{N_1-1} e^{-2\pi i(\ell_j + \phi)k_j/N} \right) \left( \prod_{j=1}^{M_2} \frac{1}{QN} \sum_{n_j=0}^{QN-1} e^{-2\pi i n_j K_j/QN} \right) \quad (3.5d)$$

with

$$0 \leq k_j \leq N-1 \quad 0 \leq K_j \leq QN-1. \quad (3.5e)$$

### 3.2. Exact wavefunctions of the two-component Schrödinger operator

Krivnov and Ovchinnikov [8] have shown that for a certain choice of the couplings, the ground-state wavefunction of (3.1) is given by

$$\psi_0 = \left| \prod_{1 \leq j < k \leq M_1} V(x_k - x_j) \prod_{1 \leq j < k \leq M_2} (V(y_k - y_j))^{Q^2} \prod_{j=1}^{M_1} \prod_{k=1}^{M_2} (V(y_k - x_j))^Q \right|^m. \quad (3.6)$$

The required couplings are

$$g_1 = 2m(m-1) \quad g_2 = 2mQ(mQ^2-1) \quad g_{12} = m(1+Q)(mQ-1) \quad (3.7)$$

and the corresponding ground state energy is

$$E_0 = \left( \frac{\pi}{N} \right)^2 m^2 \left( \frac{M_1(M_1^2-1)}{3} + Q^3 \frac{M_2(M_2^2-1)}{3} + \frac{2}{3} Q M_1 M_2 (M_1 - 1 + Q(M_2 - 1)) \right). \quad (3.8)$$

Excited states of the Hamiltonian (3.1) were not discussed in [8]. They can be investigated by seeking a trial solution of the Schrödinger equation of the form  $\phi_0 \psi$ , which gives the eigenvalue equation

$$\left[ - \sum_{j=1}^{M_1} \frac{\partial^2}{\partial(x_j)^2} - \frac{1}{Q} \sum_{j=1}^{M_2} \frac{\partial^2}{\partial(y_j)^2} - \left( \frac{2\pi m}{N} \right) \left\{ \sum_{1 \leq j < k \leq M_1} \cot \pi \frac{(x_k - x_j)}{N} \right. \right. \\ \times \left( \frac{\partial}{\partial x_k} - \frac{\partial}{\partial x_j} \right) - Q \sum_{1 \leq j < k \leq M_2} \cot \pi \frac{(y_k - y_j)}{N} \left( \frac{\partial}{\partial y_k} - \frac{\partial}{\partial y_j} \right) \\ \left. \left. - Q \sum_{j=1}^{M_1} \sum_{k=1}^{M_2} \cot \pi \frac{(x_j - y_k)}{N} \left( \frac{\partial}{\partial x_j} - \frac{1}{Q} \frac{\partial}{\partial y_k} \right) \right\} \right] \phi = (E - E_0) \phi. \quad (3.9)$$

We have been unable to solve (3.9) in general. However, we observe that

$$\phi_J = \prod_{j=1}^{M_1} e^{2\pi i J x_j / N} \prod_{k=1}^{M_2} e^{2\pi i J Q y_k / N} \tag{3.10}$$

is a solution with corresponding energy eigenvalue

$$E = E_0 + \left(\frac{2\pi J}{N}\right)^2 (M_1 + Q M_2). \tag{3.11}$$

We are also interested in the circumstance under which the Fourier expansion of  $\psi_0 \phi_J$ , with  $y_j$  replaced by  $y_j / Q$ ,  $j = 1, \dots, M_2$ , has components entirely within a single Brillouin zone, so that

$$\psi_0 \phi_J = \sum_{j_1=0}^{N-1} \dots \sum_{j_{M_1}=0}^{N-1} \sum_{k_1=0}^{QN-1} \dots \sum_{k_{M_2}=0}^{QN-1} a_{j_1 \dots k_{M_2}} \prod_{l=1}^{M_1} e^{2\pi i j_l x_l / N} \prod_{l'=1}^{M_2} e^{2\pi i k_{l'} y_{l'} / QN}. \tag{3.12}$$

From (3.6), (2.2c) and (3.10) this requires  $m, mQ$  and  $mQ^2$  to be even positive integers and

$$0 \leq -\frac{1}{2}m(M_1 - 1) - \frac{1}{2}mQM_2 + J \tag{3.13a}$$

$$N - 1 \geq \frac{1}{2}m(M_1 - 1) + \frac{1}{2}mQM_2 + J \tag{3.13b}$$

for the variables  $x_j$  and

$$0 \leq -\frac{1}{2}mQ^2(M_2 - 1) - \frac{1}{2}mQM_1 + JQ \tag{3.13c}$$

$$QN - 1 \geq \frac{1}{2}mQ^2(M_2 - 1) + \frac{1}{2}mQM_1 + JQ \tag{3.13d}$$

for the variables  $y_j$ .

### 3.3. Exact eigenstates for the two-component Bose lattice gas

We seek an equation with structure identical to (3.5a). To obtain such an equation, consider the Schrödinger equation with Hamiltonian (3.1), wavefunction

$$\psi_{J-N/2}(x_1, \dots, x_{M_1}; y_1, \dots, y_{M_2}) := \psi_0 \phi_{J-N/2} \tag{3.14}$$

where  $\psi_0$  is given by (3.6) and  $\phi_J$  by (3.10), and corresponding energy eigenvalue (3.11) (with  $J$  replaced by  $J - N/2$ ), to be denoted  $E_{J-N/2}$ . Take the Fourier transform with the Fourier variables  $k_j$  replaced by  $k_j - N/2$ ,  $j = 1, \dots, M_1$ ,

and  $K_j$  replaced by  $K_j - NQ/2, j = 1, \dots, M_2$ . After then changing variables  $y_j \mapsto y_j/Q, j = 1, \dots, M_2$ , this gives

$$\begin{aligned} & \left(\frac{2\pi}{N}\right)^2 \left( \sum_{j=1}^{M_1} (k_j - N/2)^2 + \frac{1}{Q} \sum_{j=1}^{M_2} (K_j - QN/2)^2 \right) \\ & \quad \times \tilde{\psi}_J(k_1, \dots, k_{M_1}; K_1, \dots, K_{M_2}) + I(k_1, \dots, k_{M_1}; K_1, \dots, K_{M_2}) \\ & = E_{J-N/2} \tilde{\psi}_J(k_1, \dots, k_{M_1}; K_1, \dots, K_{M_2}) \end{aligned} \tag{3.15a}$$

where

$$\tilde{\psi}_J = F[\psi_J(x_1, \dots, x_{M_1}; y_1/Q, \dots, y_{M_2}/Q)] \tag{3.15b}$$

$$\begin{aligned} I = F \left[ g_{11} \sum_{1 \leq j < k \leq M_1} V(x_k - x_j) + g_{22} \sum_{1 \leq j < k \leq M_2} V((y_k - y_j)/Q) \right. \\ \left. + g_{12} \sum_{j=1}^{M_1} \sum_{k=1}^{M_2} V(x_j - y_k/Q) \right] \psi_J(x_1, \dots, x_{M_1}; y_1/Q, \dots, y_{M_2}/Q) \end{aligned} \tag{3.15c}$$

and the Fourier operator  $F$  is defined as

$$F := \left( \prod_{j=1}^{M_1} \frac{1}{N} \int_0^N dx_j e^{-2\pi i x_j k_j/N} \right) \left( \prod_{j=1}^{M_2} \frac{1}{QN} \int_0^{QN} dy_j e^{-2\pi i y_j K_j/QN} \right). \tag{3.15d}$$

Now suppose the conditions of the sentence after (3.12) hold so that the wavefunction  $\psi_J$  in (3.15) has a Fourier expansion of the form (3.12), and define the unknown eigenstate in (3.5) by

$$\psi(\ell_1, \dots, \ell_{M_1}; n_1, \dots, n_{M_2}) = \psi_J(\ell_1 + \phi, \dots, \ell_{M_1} + \phi; n_1/Q, \dots, n_{M_2}/Q). \tag{3.16}$$

Then, following the reasoning of the paragraph after (2.23), we conclude that the equations (3.15a) and (3.5a) are identical, with the further identifications

$$\Delta_{11} = g_{11}/4 \quad \Delta_{22} = g_{22}/4 \quad \Delta_{12} = g_{12}/4 \tag{3.17a}$$

$$\lambda = \frac{E_{J-N/2}}{4} - \frac{1}{12} \left(\frac{\pi}{N}\right)^2 [M_1(N^2 + 2) + M_2((QN)^2 + 2)/Q]. \tag{3.17b}$$

In summary, we have thus provided a family of exact eigenstates (3.16) and corresponding energies (3.17b) of the two-component Bose lattice gas Hamiltonian (3.2), when the couplings are related by (3.17a) and (3.7), the parameters  $m, mQ$  and  $mQ^2$  are positive even integers, and the further restrictions (3.13) are obeyed.

3.4. Exact eigenstates for a two-component Heisenberg chain with  $1/r^2$  exchange

Crucial to the mapping between the Heisenberg chain Hamiltonian (1.1) and the Bose lattice gas Hamiltonian (2.2a) is the indistinguishability of each lattice site. This then allows the quantity

$$\sum_{1 \leq n < n' \leq N} V(n - n')(a_n^+ a_n + a_{n'}^+ a_{n'}) \tag{3.18}$$

to be evaluated in terms of the eigenvalues of the operator (2.3).

In general, this property is not true in the two-component case. Lattice sites on individual sublattices are not indistinguishable, so it is not possible to evaluate

$$\sum_{\ell=1}^N \sum_{n=1}^{QN} V(\ell + \phi - n/Q)(a_{\ell}^+ a_{\ell} + \alpha_n^+ a_n) \tag{3.19}$$

in terms of the eigenvalues of the operators (3.3). The only exception is the case

$$Q = 2 \quad \text{and} \quad \phi = \frac{1}{4} \tag{3.20}$$

when, as seen in figure 1, lattice sites on both sublattices are indistinguishable (the same is true of the case  $Q = \frac{1}{2}, \phi = \frac{1}{2}$ ; this is equivalent to (3.20) with species 1 and species 2 interchanged).

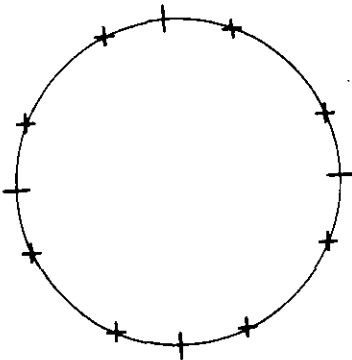


Figure 1. The two-component lattice  $Q = 2$  and  $\phi = \frac{1}{4}$  for  $N = 4$ . One sublattice is marked by dashes and the other by crosses.

The special case (3.20) of the two-component Bose lattice gas Hamiltonian (3.2) can be mapped to a two-component Heisenberg chain Hamiltonian. After introducing the spin operators (2.1) (and a second species  $s_j^{(x)}, s_j^{(y)}, s_j^{(z)}$  corresponding to the operators  $\alpha_j$  and  $\alpha_j^+$ ) the Hamiltonian (3.2) assumes the form

$$H + C \tag{3.21}$$

where

$$H = \sum_{1 \leq \ell < \ell' \leq N} V(\ell - \ell')(S_{\ell}^x S_{\ell'}^x + S_{\ell}^y S_{\ell'}^y + \Delta_{11} S_{\ell}^z S_{\ell'}^z)$$

$$\begin{aligned}
& + \frac{1}{2} \sum_{1 \leq n < n' \leq 2N} V\left(\frac{1}{2}(n - n')\right) (s_n^x s_{n'}^x + s_n^y s_{n'}^y + \Delta_{22} s_n^z s_{n'}^z) \\
& + \Delta_{12} \sum_{\ell=1}^N \sum_{n=1}^{2N} V\left(\ell + \frac{1}{4} - \frac{1}{2}n\right) S_\ell^z s_n^z
\end{aligned} \tag{3.22}$$

and

$$\begin{aligned}
C & = \frac{1}{2} \Delta_{11} C_{11} \left(M_1 - \frac{1}{4}N\right) + \frac{1}{2} \Delta_{22} C_{22} \left(M_2 - \frac{1}{2}N\right) \\
& + \frac{1}{2} \Delta_{12} \left(M_2 C_{21} + C_{12} \left(M_1 - \frac{1}{2}N\right)\right)
\end{aligned} \tag{3.23a}$$

with

$$C_{11} = \sum_{j=1}^{N-1} V(j) = \frac{1}{3} \left(\frac{\pi}{N}\right)^2 (N^2 - 1) \tag{3.23b}$$

$$C_{22} = \sum_{j=1}^{2N-1} V(j/2) = \frac{1}{3} \left(\frac{\pi}{N}\right)^2 (4N^2 - 1)$$

$$C_{12} = \sum_{j=0}^{2N-1} V\left(\frac{1}{4} - \frac{1}{2}j\right) = 4\pi^2 \tag{3.23c}$$

$$C_{21} = \sum_{j=0}^{N-1} V\left(j + \frac{1}{4}\right) = 2\pi^2.$$

The summations (3.23b) and (3.23c) follow from the identities given in the appendix.

From the results of the above section we have that if

$$\Delta_{11} = \frac{1}{2}m(m-1) \quad \Delta_{22} = m(4m-1) \quad \Delta_{12} = \frac{1}{2}m(2m-1) \tag{3.24}$$

$m$  is an even positive integer and the conditions (3.13) with  $Q = 2$  are obeyed, then the function

$$\begin{aligned}
& \prod_{j=1}^{M_1} e^{2\pi i J \ell_j / N} \prod_{j=1}^{M_2} e^{2\pi i J n_j / N} \left[ \prod_{1 \leq j < k \leq M_1} \sin \frac{\pi}{N} (\ell_k - \ell_j) \right. \\
& \left. \times \prod_{1 \leq j < k \leq M_2} \sin^4 \frac{\pi}{2N} (n_k - n_j) \prod_{j=1}^{M_1} \prod_{k=1}^{M_2} \sin^2 \frac{\pi}{N} \left(\ell_j + \frac{1}{4} - \frac{1}{2}n_k\right) \right]^m
\end{aligned} \tag{3.25}$$

is an eigenstate of (3.22) with energy

$$\begin{aligned}
& \left(\frac{\pi}{N}\right)^2 m^2 \left( \frac{M_1(M_1^2 - 1)}{3} + \frac{8M_2(M_2^2 - 1)}{3} + \frac{4}{3} M_1 M_2 (M_1 - 1 + 2(M_2 - 1)) \right) \\
& + \left( \frac{2\pi(J - N/2)}{N} \right)^2 (M_1 + 2M_2) \\
& - \frac{1}{2} \left(\frac{\pi}{N}\right)^2 [M_1(N^2 + 2) + M_2(2N^2 + 1)] - C
\end{aligned} \tag{3.26}$$

where  $C$  is given by (3.23) and  $M_1$  and  $M_2$  are eigenvalues of the operators (3.3).

**Acknowledgment**

This work was supported by the Australian Research Council.

**Appendix**

Here we will provide the evaluations of the summations

$$S(k) := \sum_{m=1}^{M-1} \frac{e^{2\pi i m k / M}}{\sin^2 \pi m / M} \tag{A1}$$

and

$$S_1(k, \phi) := \sum_{m=0}^{M-1} \frac{e^{2\pi i m k / M}}{\sin^2 \pi(m + \phi) / M} \tag{A2}$$

where  $k$  is an integer within the range  $0 \leq k \leq M - 1$  and  $\phi$  is any non-integer complex number. We note that  $S(k)$  can be deduced from  $S_1(k, \phi)$  via the formula

$$S(k) = \lim_{\phi \rightarrow 0} \left( S_1(k, \phi) - \frac{1}{\sin^2 \pi \phi M} \right). \tag{A3}$$

To evaluate  $S_1(k, \phi)$  we note that

$$\frac{1}{\sin^2 \pi(m + \phi) / M} = -4 \sum_{p=0}^{\infty} p e^{-2\pi i(m+\phi)p / M} \quad \text{Im } \phi < 0. \tag{A4}$$

Replacing the summation over  $p$  by a summation over  $p' + sM$ ,  $0 \leq p' < \infty$ ,  $0 \leq s \leq M - 1$ , allows the summation over  $p'$  to be performed. This procedure gives

$$\begin{aligned} \frac{1}{\sin^2 \pi(m + \phi) / M} &= -\frac{4}{1 - e^{-2\pi i \phi}} \sum_{s=0}^{m-1} s e^{-2\pi i s(m+\phi) / M} \\ &+ \frac{M}{\sin^2 \pi \phi} \sum_{s=0}^{M-1} e^{-2\pi i s(m+\phi) / M}. \end{aligned} \tag{A5}$$

The summation in (A2) can now be done immediately to give

$$S_1(k, \phi) = -\frac{4kM e^{-2\pi i k \phi / M}}{1 - e^{-2\pi i \phi}} + \frac{M^2 e^{-2\pi i k \phi / M}}{\sin^2 \pi \phi}. \tag{A6}$$



Taking the limit in (A3) then gives

$$S(k) = 2(k - M/2)^2 - (M^2 + 2)/6. \quad (\text{A7})$$

## References

- [1] Haldane F D M 1988 *Phys. Rev. Lett.* **60** 635
- [2] Haldane F D M 1991 *Phys. Rev. Lett.* **66** 1529
- [3] Shastry B S 1988 *Phys. Rev. Lett.* **60** 639
- [4] Inozemstev V I 1990 *J. Stat. Phys.* **59** 1143
- [5] Sutherland B 1988 *Phys. Rev. B* **38** 6689
- [6] Sutherland B 1971 *Phys. Rev. A* **4** 2019
- [7] Sutherland B 1972 *Phys. Rev. A* **5** 1372
- [8] Krivnov V Ya and Ovshinnikov A A 1982 *Theoret. Math. Phys.* **50** 100